

# A cohomological nonabelian Hodge theorem in positive characteristic

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# Abstract

By using specialization in cohomology techniques, under suitable natural coprimality conditions, we prove a cohomological Simpson correspondence between the moduli space of Higgs bundles and that of connections on a curve over an algebraically closed ground field of positive characteristic p > 0. We also prove a new *p*-multiplicative periodicity concerning the cohomology rings of Dolbeault moduli spaces of degrees differing by a factor of *p*. By coupling this *p*-periodicity in characteristic *p* with lifting/specialization techniques in mixed characteristic, we find, in arbitrary characteristic, cohomology ring isomorphisms between the cohomology rings of Dolbeault moduli spaces for different degrees coprime to the rank. Interestingly, this last result is proved by passing from positive characteristic to characteristic zero and back. The moduli spaces we work with admit certain natural morphisms, and all the cohomology ring isomorphisms we find are filtered isomorphisms for the resulting perverse Leray filtrations.

# 1. Introduction

Let C be a connected projective nonsingular curve over the complex numbers. The nonabelian Hodge theorem (a.k.a. the Simpson correspondence) [Sim94a, Sim94b] establishes that three rather different moduli spaces are canonically homeomorphic to one another: the de Rham moduli space  $M_{dR}$  of rank r connections on C, the Dolbeault moduli space  $M_{Dol}$  of rank r and degree zero Higgs bundles on C and the Betti moduli space  $M_B$  of representations of the fundamental group of C into  $GL(r, \mathbb{C})$ . There is also the Hodge moduli space  $M_{Hod}$  of t-connections [Sim97] that in some sense subsumes  $M_{Dol}$  and  $M_{dR}$ . For the variant concerning nonsingular moduli for bundles of (nonzero) degree coprime to the rank, see [HT04]. For a brief summary concerning the Hodge, Dolbeault and de Rham moduli spaces, see § 1.3.

In this paper, we also work over an algebraically closed ground field of positive characteristic, where, even though many beautiful results are available, the situation is less clear. Since there seems to be no Betti picture that fits well with a possible Simpson correspondence, in this paper, by Simpson correspondence in characteristic p > 0, we mean some kind of relation between Higgs bundles (Dolbeault picture) and connections (de Rham picture).

Keywords: Higgs bundles, perverse filtrations.

Received 6 July 2021, accepted in final form 21 February 2022.

<sup>2020</sup> Mathematics Subject Classification 14D20.

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M. A. de Cataldo is partially supported by NSF grant DMS 1901975 and by a Simons Fellowship in Mathematics. S. Zhang is partially supported by NSF grant DMS 1901975.

A. Ogus and V. Vologodsky [OV07, § 4] establish, among other things, a Simpson correspondence between the stack of Higgs bundles with nilpotent Higgs field for the Frobenius twist  $C^{(1)}$  of the curve C and the stack of connections on the curve C with nilpotent p-curvature tensor. M. Groechenig [Gro16, Theorem 3.29, Lemma 3.46] proves that there is a pair of morphisms  $M_{\text{Dol}}(C^{(1)}) \rightarrow A(C^{(1)}) \leftarrow M_{\text{dR}}(C)$  that are étale-locally equivalent over the Hitchin base  $A(C^{(1)})$  (§ 1.3), both for the coarse moduli spaces and for the stacks. T.-H. Chen and X. Zhu [CZ15, Theorem 1.2] prove an analogous result at the stack level, for arbitrary reductive groups in place of the general linear group. The reader can also consult [LSZ19] for generalizations of the isomorphism in [OV07] to the study of Higgs-de Rham flows for schemes in positive and mixed characteristic. One recovers the aforementioned nilpotent Simpson correspondence in characteristic p > 0 in [OV07], by taking the fibers of the pair of morphisms over the origin in  $A(C^{(1)})$ . More generally, we get a kind of Simpson correspondence: for every closed point in  $A(C^{(1)})$ , the two fibers of the morphisms  $M_{\text{Dol}}(C^{(1)}) \rightarrow A(C^{(1)}) \leftarrow M_{\text{dR}}(C)$  are noncanonically isomorphic varieties and thus have isomorphic étale cohomology rings. Note that these results relate Higgs bundles of degree d on  $C^{(1)}$  to connections of degree dp on C.

None of these results seems to imply a global statement concerning (the cohomology of) the Dolbeault and the de Rham moduli spaces. In short, it seems that we are still missing a (cohomological) global Simpson correspondence in positive characteristic.

In this paper, we prove such a new cohomological Simpson correspondence result for curves over an algebraically closed field of positive characteristic p > 0, as well as a series of new allied results in arbitrary characteristics. The methods we use center on the use of vanishing cycles and of the specialization morphism in equal and in mixed characteristic. In order to use these techniques, we need to establish the smoothness of certain morphisms and the properness of certain other morphisms. Once this is done, we still need to come to terms with the fact that the specialization morphisms may fail to be defined because the moduli spaces we work with are not proper over the ground field. While this issue is circumvented in the proofs of the results in § 2, it is not in the proofs of the results in § 3, where we use the compactification results of [dCZ22] and their application to specialization morphisms.

Let us describe the main results of this paper. First of all, all the cohomology rings we deal with carry natural filtrations, called perverse Leray filtrations, associated with the various morphisms—Hitchin, de Rham–Hitchin, Hodge–Hitchin ( $\S1.3$ )—exiting these moduli space. In what follows, we omit these filtrations from the notation.

Let C/k be a nonsingular connected projective curve over an algebraically closed field of characteristic  $p \ge 0$ . Let  $\ell$  be a prime, invertible in the ground field. Since the rank is fixed in what follows, we drop it from the notation.

Theorem 2.1 (Cohomological Simpson correspondence,  $\operatorname{char}(k) = p > 0$ , I) and its refinement Theorem 3.6 (Cohomological Simpson correspondence,  $\operatorname{char}(k) = p > 0$ , II). Let p > 0. We work under natural assumptions on the rank r and degree d of the vector bundles involved and on the characteristic p: namely,  $d = \overline{d}p$  is a multiple of the characteristic, and  $\operatorname{gcd}(r, d) = 1$ . Note that then  $\operatorname{gcd}(r, p) = 1$ . The first condition is to have nonempty de Rham space/stack; the second one is to have nonsingular moduli spaces. Then we prove that there is a canonical filtered isomorphism between the corresponding étale cohomology rings

$$H^*(M_{\text{Dol}}(C;d),\overline{\mathbb{Q}}_{\ell}) \simeq H^*(M_{\text{dR}}(C;d),\overline{\mathbb{Q}}_{\ell}).$$
(1)

Unlike [OV07, CZ15, Gro16], equation (1) relates the étale cohomology rings of the Dolbeault

and de Rham moduli spaces for the same curve C and the same degree. While the Frobenius twist  $C^{(1)}$  does not appear in the statement of (1), it plays a key role in the proof.

Theorem 2.4 (The cohomology ring of  $N_{dR}$ ). Let p > 0, and assume the same conditions on r and d seen above:  $d = \overline{d}p$  and gcd(r, d) = 1. We use (22) from the proof of Theorem 2.1 to prove that there is a canonical filtered isomorphism of cohomology rings

$$H^*(M_{\mathrm{dR}}(C;d),\overline{\mathbb{Q}}_\ell) \simeq H^*(N_{\mathrm{dR}}(C;d),\overline{\mathbb{Q}}_\ell), \qquad (2)$$

where  $N_{dR}$  is the subspace of stable connection with nilpotent *p*-curvature, that is, the fiber over the origin of the de Rham-Hitchin morphism  $h_{dR}: M_{dR} \to A(C^{(1)})$  (see § 1.3). The corresponding fact for  $M_{Dol}$  and the fiber  $N_{Dol}$  is well known and valid without any assumptions on rank and degree, and it can be proved by using the theory of weights jointly with the classical contracting  $\mathbb{G}_m$ -action on the  $\mathbb{G}_m$ -equivariant and proper Hitchin morphism  $h_{Dol}: M_{Dol} \to A(C)$ . The surprising aspect of (2) is that there is no known  $\mathbb{G}_m$ -action on  $M_{dR}$ .

Theorem 2.5 (*p*-Multiplicative periodicity with Frobenius twists). Let p > 0, and assume the same conditions on r and d seen above:  $d = \overline{d}p$  and gcd(r, d) = 1. This theorem expresses a new periodicity feature concerning the cohomology rings of Dolbeault moduli spaces for degrees that differ by a multiple a power of the characteristic p > 0; namely, there is a canonical filtered isomorphism of cohomology rings

$$H^*(M_{\text{Dol}}(C;\overline{d}),\overline{\mathbb{Q}}_{\ell}) \simeq H^*(M_{\text{Dol}}(C^{(-m)};\overline{d}p^m),\overline{\mathbb{Q}}_{\ell}), \qquad (3)$$

where  $m \ge 0$  and  $C^{(-m)}$  is the (-m)th Frobenius twist of C, that is, the base change of C/k via the *m*th power  $fr_k^{-m} \colon k \xrightarrow{\sim} k, a \mapsto a^{p^{-m}}$ , of the inverse of the absolute Frobenius automorphism  $fr_k$ .

Theorem 3.9 (Different curves, same degree). Let  $p \ge 0$ , and let gcd(r, d) = 1. We do not assume that the degree is a multiple of p. We prove that the cohomology rings of the Dolbeault moduli spaces of two curves  $C_i$  of the same genus are noncanonically filtered isomorphic:

$$H^*(M_{\text{Dol}}(C_1; d)) \simeq H^*(M_{\text{Dol}}(C_2; d)).$$

Over the complex numbers, the statement without the filtrations is an easy consequence of the fact that the two Dolbeault moduli spaces are diffeomorphic to the (common) Betti moduli space; the filtered statement is proved in [dCM20].

Theorem 3.11 (*p*-Multiplicative periodicity without Frobenius twists). Let p > 0, and assume the same conditions on r and d seen above:  $d = \overline{d}p$  and gcd(r, d) = 1. We prove a noncanonical analog of (3), with the Frobenius twist  $C^{(-m)}$  replaced by the original curve C (or, in fact, by any curve of the same genus, in view of Theorem 3.9):

$$H^*(M_{\text{Dol}}(C;\overline{d}),\overline{\mathbb{Q}}_\ell) \simeq H^*(M_{\text{Dol}}(C;\overline{d}p^m),\overline{\mathbb{Q}}_\ell).$$
(4)

Theorem 3.12 (Same curve, different degrees; char(k) = 0). Here, p = 0. Let d, d' be degrees coprime to the rank r. We prove that the cohomology rings of the Dolbeault moduli spaces in degrees d, d' for a curve C are filtered isomorphic:

$$H^*(M_{\text{Dol}}(C,d),\overline{\mathbb{Q}}_{\ell}) \simeq H^*(M_{\text{Dol}}(C,d'),\overline{\mathbb{Q}}_{\ell}).$$
(5)

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Over the complex numbers, the statement without the filtrations is a consequence of the fact that the two Dolbeault moduli spaces are diffeomorphic to their Betti counterparts and that, in turn, these are Galois-conjugate. The resulting "transcendental" isomorphism differs from the isomorphism in Theorem 3.12. Presently, it is not known how to compare the perverse Leray filtrations under the "transcendental" isomorphism. *Added in revision:* 1) This comparison is the subject of [dCMSZ21]: the two match. 2) In the recent paper by T. Kinjo and N. Koseki [KK21, Theorem 1.1], an isomorphism of the form (5) is obtained by a method that differs from ours.

Theorem 3.13 (Same curve, different degrees; char(k) = p > 0). Here, p > 0. Let d, d' be degrees; coprime to the rank r and assume p > r. Then we prove the statement analogous to Theorem 3.12.

We want to emphasize the following amusing fact: Theorem 3.11 (a result in positive characteristic) is used to prove Theorem 3.12 (a result in characteristic zero); in turn, this latter result is used to prove Theorem 3.13 (a result in positive characteristic).

## 1.1 Notation and preliminaries

The schemes we work with. We fix a base ring J that is either a field or a discrete valuation ring (DVR), possibly of mixed characteristic (0, p > 0). We work with separated schemes of finite type over J and with J-morphisms that are separated and of finite type. The term variety is reserved to schemes as above when the base is a field.

Constructible derived categories and perverse t-structures over the DVR. Let  $\ell$  be a prime number invertible in J. We employ the usual formalism of the corresponding "derived" categories  $D_c^b(-, \overline{\mathbb{Q}}_{\ell})$  of bounded constructible "complexes" of  $\overline{\mathbb{Q}}_{\ell}$ -adic sheaves endowed with the appropriate version of the middle perversity t-structure: the classical one if J is a field and the rectified one if J is a DVR as above. When working over a field with the usual six functors and the perverse t-structure, the references [Eke90, Theorem 6.3] and [BBD82] are sufficient for our purposes. When working over a DVR as above, we need to complement these references so that we can work with nearby/vanishing cycles functors and their t-exactness properties for the rectified perverse t-structure. For a discussion and additional references, see [dCZ22, § 5.2].

The perverse Leray filtration. Étale cohomology groups are taken only for varieties over algebraically closed fields J = k. More often than not, we drop "étale." Let  $f: X \to Y$  be a k-morphism, and let  $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ . We denote the functor  $Rf_*$  simply by  $f_*$ ; the derived direct images are denoted by  $R^{\bullet}f_*$ , for  $\bullet \in \mathbb{Z}$ . We denote the perverse truncation functors by  ${}^{\mathfrak{p}}\tau_{\leq \bullet}$ , for  $\bullet \in \mathbb{Z}$ . The increasing perverse Leray filtration  $P^f_{\bullet}$  on  $H^*(X, K)$  is defined by setting, for every  $\bullet, \star \in \mathbb{Z}$ ,

$$P^{f}_{\bullet}H^{\star}(X,K) := \operatorname{Im}\{H^{\star}(Y, \,^{\mathfrak{p}}_{\tau \leqslant \bullet}Rf_{*}K) \to H^{\star}(Y,Rf_{*}K) = H^{\star}(X,K)\}.$$

$$(6)$$

Let  $f: X \to Y$  and  $g: Y \to Z$  be morphisms of k-varieties. If g is finite, then  $g_*$  is t-exact (hence, being cohomological, exact on the category of perverse sheaves), so that

$$P_{\bullet}^{g\circ f}H^{\star}(X,K) = P_{\bullet}^{f}H^{\star}(X,K).$$
<sup>(7)</sup>

*Étale cohomology rings.* When working with separated schemes of finite type (varieties) over an algebraically closed field k of positive characteristic p > 0, we fix any other prime  $\ell \neq p$ .

The graded étale cohomology groups  $H^*(-, \overline{\mathbb{Q}}_{\ell})$  of such a variety form a unital, associative, graded-commutative  $\overline{\mathbb{Q}}_{\ell}$ -algebra for the cup product operation. A graded morphism between the graded étale cohomology groups of two varieties preserving these structures is simply called a morphism of cohomology rings. Of course, pull-backs via morphisms are examples. In this paper, we find isomorphisms of cohomology rings, with additional compatibilities, that do not arise from morphisms.

# 1.2 Background on vanishing/nearby cycles and specialization in cohomology

We briefly recall the general setup for the formalism of nearby-vanishing cycles using strictly Henselian traits; see [DRR<sup>+</sup>72, Ill94] and [Eke90, §6, Remark, p. 214]. Caveat: There are several distinct and all well-established ways to denote nearby/vanishing cycles in the literature. Our notation  $\phi$  for the vanishing cycle differs by a shift (our  $\phi$ [1] is their  $\phi$ ) with respect to the given references. Our current notation makes  $\phi$  and  $\psi$ [-1] *t*-exact functors, and is in accordance with [dCa22, dCZ22], as well as with other occurrences in the literature.

Strictly Henselian traits. Let  $(S, s, \eta, \overline{\eta})$  be a strictly Henselian trait together with a minimal choice of generic geometric point; that is,

- (i) S is the spectrum of a strictly Henselian DVR, hence with separably closed residue field;
- (ii)  $i: s \to S$  is the closed point (it is also a geometric point);

(iii)

$$\bar{j} \colon \bar{\eta} \longrightarrow \eta \longrightarrow S \tag{8}$$

is the generic point of S, with the associated geometric point stemming from a fixed choice of a separable closure  $k(\eta)^{\text{sep}}/k(\eta)$  of the fraction field of the Henselian ring.

The objects restricted via the base change  $i: s \to S$  are denoted by a subscript -s, and similarly for  $-\eta$  and for  $-\overline{\eta}$ .

*Vanishing/nearby cycles.* Let  $v \colon X \to S$  be a morphism of finite type. We have the distinguished triangle of functors

$$i^* \longrightarrow \psi_v \longrightarrow \phi_v[1] \leadsto ,$$

where the three functors are functors  $D_c^b(X, \overline{\mathbb{Q}}_\ell) \to D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$ . The functor  $\psi_v$  is called the nearby cycle functor, and the functor  $\phi_v$  is called the vanishing cycle functor. By restricting to  $\eta$ , we can also view the functor  $\psi_v$  as a functor  $D_c^b(X_\eta) \to D_c^b(X_s)$ . If  $\eta^* F \simeq \eta^* G$ , then  $\psi_v(F) \simeq \psi_v(G)$ , functorially.

The specialization morphism sp. For F in  $D_c^b(X)$ , we have the fundamental diagram

$$H^*(X_s, F) \xrightarrow{i^*} H^*(X, F) \xrightarrow{\overline{\eta}^*} H(X_{\overline{\eta}}, F) .$$

$$\tag{9}$$

If  $i^*$  is an isomorphism, then we define the specialization morphism by setting

$$R^{\bullet}v_*i^*F = H^{\bullet}(X_s, F) \xrightarrow{\text{sp}:=\overline{\eta}^* \circ (i^*)^{-1}} H^{\bullet}(X_{\overline{\eta}}, F) = \mathcal{H}^{\bullet}_s(\psi_v v_*F), \quad \forall \bullet \in \mathbb{Z}.$$

By the proper base change theorem, if v is proper, then  $i^*$  is an isomorphism and the specialization morphism is defined. However, it v is not proper, then  $i^*$  may fail to be an isomorphism and the

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specialization morphism may fail to be defined. The paper [dCa22] is devoted to exploring this phenomenon, and in this paper, we work in such a situation.

Remark 1.1. If the specialization morphism is defined, then it is compatible with cup products, for example, when  $F = \overline{\mathbb{Q}}_{\ell}$ . More generally, it is compatible with pairings  $F' \otimes F'' \to F$  of objects in  $D_c^b(X)$  [Ill94, § 4.3].

FACT 1.2. For the purpose of this paper, the most important properties of the vanishing cycle functors are:

- (i) If v is smooth, then  $\phi_v(\overline{\mathbb{Q}}_\ell) = 0$ ; see [DRR<sup>+</sup>72, Chapter XIII, Reformulation 2.1.5].
- (ii) If  $f: Y \to X$  is a proper morphism, and  $u: Y \to S$  and  $f_s: Y_s \to X_s$  are the resulting morphisms, then, by proper base change, we have natural isomorphisms  $\phi_v f_* = f_{s,*}\phi_u$  and  $\psi_v f_* = f_{s,*}\psi_u$ ; see [DRR<sup>+</sup>72, Chapter XIII, (2.1.7.1)].

The moduli spaces we work with are not proper over their base, so that it is not clear at the outset that the various specialization morphisms we wish to consider are even defined. In this context, we prove Proposition 3.4 for use in § 3. On the other hand, in § 2, we circumvent the direct use of these specialization morphisms; see the proof of Theorem 2.1.

# 1.3 The moduli spaces we work with

The existence, quasi-projectivity, and uniform (universal in the coprime case when not in characteristic zero) corepresentability of the moduli spaces we are about to introduce have been established by C. Simpson [Sim94a, Sim94b] for smooth projective families over a base of finite type over a ground field of characteristic zero, and over a base of finite type over a universally Japanese ring by A. Langer [Lan14, Theorem 1.1]. Recall that "universal" (respectively, "uniform") refers to the commutation of the formation of the coarse moduli space with arbitrary (respectively, flat) base change.

Base over base ring. In this paper, we only need to consider the setup of a base B that is Noetherian and of finite type over a base ring J, that is, either an algebraically closed field kor a DVR. For a more general setup and more details concerning the moduli spaces we use, see [dCZ22]. Note that for the sake of the existence of the moduli spaces, the assumption on the base has been relaxed to B being any Noetherian scheme in A. Langer's recent paper [Lan21, Theorem 1.1].

Smooth curves. In this paper, a smooth curve C/B is a smooth projective morphism  $C \to B$  with geometric fibers integral of dimension 1. If the base B = J = k is a field, then we often write C instead of C/k.

Coprimality assumption on rank, degree and characteristic of the ground field. When working with vector bundles, we denote their rank by r and their degree by d. In this paper, we always assume that they are coprime; that is, gcd(r, d) = 1. When working with the de Rham moduli space of stable (= semistable) connections on a smooth curve over an algebraically closed field of positive characteristic p > 0, we always assume, in addition, that the degree  $d = \overline{d}p$  is an integer multiple of the characteristic p; otherwise, there are no such connections. Our assumptions imply that stability coincides with semistability, thus ensuring: 1) the nonsingularity of the Hodge (t-connections), Dolbeault (Higgs bundles) and de Rham (connections) moduli spaces (cf. § 1.4);

2) that these moduli universally (instead of merely uniformly) corepresent their moduli functor [Lan14, Theorem 1.1], so that the formation of such moduli spaces commutes with arbitrary base change into the moduli space, hence in particular into B or J.

Regrettably, the coprimality assumptions rules out the important case of connections of degree zero. On the other hand, these assumptions are the most natural when dealing with nonsingular moduli spaces. While our methods require 1) and 2) above, one wonders if many of the result of this paper hold without the coprimality assumption, that is, for the possibly singular Hodge/Dolbeault/de Rham moduli spaces that arise. We are not sure what to expect in the singular case. Also note that the "p-multiplicative periodicity" results Theorems 2.5 and 3.11 express a property of the Dolbeault moduli spaces that acquires a nontrivial meaning only in nonzero degrees; the same holds for Theorems 3.12 and 3.13.

The Hodge moduli space. A t-connection on a smooth curve C/B is a triple  $(t, E, \nabla_t)$ , where t is a regular function on B, E is a vector bundle on C and  $\nabla_t : E \to E \otimes_{\mathcal{O}_C} \Omega^1_{C/B}$  is  $\mathcal{O}_B$ -linear and satisfies the twisted Leibnitz rule  $\nabla_t(f\sigma) = tdf \otimes \sigma + f\nabla_t(\sigma)$  for every local function f on C and every local section  $\sigma$  of E on C on the same open subset. There is the quasi-projective B-scheme  $M_{\text{Hod}}(C/B; r, d)$  (cf. [Lan14, Theorem 1.1]), the coarse Hodge moduli space universally corepresenting slope stable t-connections of rank r and degree d on the smooth curve C/B. It comes with a natural B-morphism of finite type to the affine line assigning t to a t-connection

$$\tau_{\mathrm{Hod}}(C/B; r, d) \colon M_{\mathrm{Hod}}(C/B; r, d) \longrightarrow \mathbb{A}^1_B.$$
(10)

Dolbeault moduli space and Hitchin morphism. By the universal corepresentability property, if we take the fiber over the origin  $0_B \to \mathbb{A}^1_B$ , then we obtain the quasi-projective B-scheme

$$M_{\text{Dol}}(C/B; r, d)$$
,

the coarse Dolbeault moduli space universally corepresenting slope stable rank r and degree d Higgs bundles, twisted by the canonical bundle, on the family of curves C/B. If B is a field, then the Dolbeault moduli space is empty if and only if the genus of the curve is zero and the rank  $r \ge 2$ ; otherwise, this moduli space is integral, nonsingular and of dimension that depends only on the rank r and genus g of the curve (cf. [Nit91, §7]):

$$\dim M_{\text{Dol}}(C, r, d) = r^2 (2g - 2) + 2.$$
(11)

Let A(C/B; r) be the vector bundle on B of rank one half the dimension (11), with fiber  $H^0(C_b, \bigoplus_{i=1}^r \omega_{C_b}^{\otimes i})$ . There is the projective and surjective Hitchin B-morphism

$$h_{\text{Dol}}(C/B; r, d) \colon M_{\text{Dol}}(C/B; r, d) \longrightarrow A(C/B; r),$$
 (12)

assigning to a Higgs bundle the characteristic polynomial of its Higgs field. For the projectivity of the Hitchin morphism over a base, see [dCZ22, Theorem 2.18].

The Hitchin base. The B-scheme A(C/B; r) is sometimes called the Hitchin base, or the space of characteristic polynomials of rank r Higgs fields, or the space of degree r spectral curves over C/B.

De Rham moduli space and de Rham-Hitchin morphism. If we take the fiber of (10) over  $1_B \to \mathbb{A}^1_B$ , then we obtain the quasi-projective B-scheme

$$M_{\mathrm{dR}}(C/B; r, d)$$
,

the coarse de Rham moduli space universally corepresenting slope rank r and degree d stable connections on the family of curves C/B.

If J = k is an algebraically closed field of characteristic zero, then the de Rham moduli space is nonempty if and only if d = 0.

If J = k is an algebraically closed field of positive characteristic p, then the de Rham moduli space is nonempty if and only if  $d = \overline{d}p$  is an integer multiple of p (recall that this is part of our assumptions on rank, degree and characteristic); see [BS06, Proposition 3.1]. In this case, it is shown in Lemma 1.4 that the de Rham moduli space is integral, nonsingular, of the same dimension (11) as the Dolbeault moduli space for the same rank and degree. In this case, we also have the projective and surjective de Rham–Hitchin *B*-morphism

$$h_{\mathrm{dR}}(C/B; r, \overline{d}p): M_{\mathrm{dR}}(C/B; r, \overline{d}p) \longrightarrow A(C^{(B)}/B; r),$$

where  $C^{(B)}/B$  is the base change of C/B via the absolute Frobenius endomorphism  $fr_B: B \to B$ (absolute Frobenius for B: identity of topological space; functions raised to the pth power). The de Rham–Hitchin morphism is defined in [Gro16, Definition 3.16]. It is shown to be proper in [Gro16, Corollary 3.47], thus projective in view of the quasi-projectivity at the source. For every closed point  $b \in B$ , the fiber  $(C^{(B)}/B)_b = (C_b)^{(1)} =: \kappa(b) \times_{\kappa(b), fr_{\kappa(b)}} C$  is the Frobenius twist of the curve  $C/\kappa(b)$ , that is, the base change of  $C/\kappa(b)$  via the absolute Frobenius automorphism  $fr_{\kappa(b)}$ of  $\kappa(b)$ . The fiber at  $b \in B$  of the vector bundle  $A(C^{(B)}/B; r)$  is given by  $\bigoplus_{i=1}^r H^0(C_b^{(1)}, \omega_{C_b^{(1)}}^{\otimes i})$ .

Hodge–Hitchin morphism (char(k) = p > 0). Let J = k be an algebraically closed field of positive characteristic p > 0. Y. Lazslo and C. Pauly [LP01] (see also [dCZ22]) have constructed a natural factorization of the morphism  $\tau_{\text{Hod}}$  (10)

$$\tau_{\mathrm{Hod}}(C/B; r, d) \colon M_{\mathrm{Hod}}(C/B; r, d) \xrightarrow{h_{\mathrm{Hod}}(C/B; r, d)} A(X^{(B)}/B; r) \times_B \mathbb{A}^1_B \xrightarrow{\mathrm{pr}_2} \mathbb{A}^1_B.$$
(13)

We call the quasi-projective *B*-morphism  $h_{\text{Hod}}(C/B; r, d)$  the Hodge–Hitchin morphism. It assigns to a *t*-connection on a curve *C* the characteristic polynomial of its *p*-curvature. The *p*curvature is a Higgs field on the same underlying vector bundle on the curve *C*, but for the *p*th power of the canonical line bundle. The key observation is that this characteristic polynomial is the pull-back via the relative Frobenius morphism  $\text{Fr}_C \colon C \to C^{(1)}$  of a uniquely determined characteristic polynomial on  $C^{(1)}$ .

If we specialize  $h_{\text{Hod}}(C/B; r, d)$  at  $1_B$ , then we obtain the de Rham-Hitchin morphism

$$h_{\mathrm{dR}}(C/B;r,d) := h_{\mathrm{Hod}}(C/B;r,d)_{1_B} \colon M_{\mathrm{dR}}(C/B;r,d) \longrightarrow A(X^{(B)}/B;r) .$$
(14)

If we specialize  $h_{\text{Hod}}(C/B; r, d)$  at  $0_B$ , then we obtain the classical Hitchin morphism postcomposed with the Frobenius relative to B (see [dCZ22])

$$h_{\mathrm{Hod}}(C/B;r,d)_{0_B} \colon M_{\mathrm{Dol}}(C/B;r,d) \xrightarrow{h_{\mathrm{Dol}}(C/B;r,d)} A(C/B;r) \xrightarrow{\mathrm{Fr}_{A(C/B;r)/B}} A(X^{(B)}/B;r) .$$
(15)

 $\mathbb{G}_m$ -actions and equivariance. The group scheme  $\mathbb{G}_{m,B}$  acts on the Hodge moduli space by weight 1 dilatation on the t-connections:  $\lambda \cdot \nabla_t := \nabla_{\lambda t}$ , and similarly on  $\mathbb{A}^1_B$ . The morphism  $\tau$  (10) is  $\mathbb{G}_{m,B}$ -equivariant for these actions. Moreover, the pre-image of  $\mathbb{G}_{m,B} \subseteq \mathbb{A}^1_B$  is canonically and  $\mathbb{G}_{m,B}$ -equivariantly a fiber product over B of the de Rham moduli space times  $\mathbb{G}_{m,B}$ ; that is, we have (see [dCZ22])

$$\tau^{-1}(\mathbb{G}_{m,B}) \simeq M_{\mathrm{dR}}(C/B) \times_B \mathbb{G}_{m,B}.$$
(16)

If J = k is an algebraically closed field of positive characteristic p > 0, then the group scheme  $\mathbb{G}_{m,B}$  acts on  $A(C^{(B)}/B;r) \times_B \mathbb{A}^1_B$ : by weight 1 dilations on  $\mathbb{A}^1_B$  and by weight *ip* dilations on each term  $H^0(C_b^{(1)}, \omega_{C^{(1)}}^{\otimes i})$ .

If J is arbitrary, then the group scheme  $\mathbb{G}_{m,B}$  acts on  $A(C/B; r) \times_B \mathbb{A}^1_B$  in a similar way, but by weight *i* dilations on each term  $H^0(C_b, \omega_{C_b}^{\otimes i})$ .

All the morphisms appearing in (13), (14) and (15) are  $\mathbb{G}_{m,B}$ -equivariant for specified actions. Moreover, the trivialization (16) extends to an evident  $\mathbb{G}_{m,B}$ -equivariant trivialization of (13) over  $\mathbb{G}_{m,B} \subseteq \mathbb{A}^1_B$ , and, in particular, we have a natural  $\mathbb{G}_{m,B}$ -equivariant identification

$$h_{\text{Hod}}|\mathbb{G}_{m,B} = h_{\text{dR}} \times_B \text{Id}_{\mathbb{G}_{m,B}}.$$
(17)

Even without the coprimality assumption, the following properness statement is proved in [dCZ22, Theorem 2.13(2)], and it can also be seen as a consequence of what is stated in [Lan14, § 4.5, top of p. 531]. We thank A. Langer for providing us with a proof in a private communication (added in revision: A. Langer's communication now appears in [Lan21, Theorem 1.3]). This properness result plays an essential role in this paper. An alternative proof of this properness under our coprimality assumptions is given in Proposition 1.11 which, in turn, is based on the ad hoc criterion Proposition 1.9.

THEOREM 1.3. The Hodge–Hitchin morphism  $h_{\text{Hod}}$  (13) is proper, in fact projective.

## 1.4 Smoothness of moduli spaces

In this section, we place ourselves in the following special case of the setup in §1.3: C = C/k is a smooth curve over an algebraically closed field k of positive characteristic p, the degree  $d = \overline{d}p$ is an integer multiple of the characteristic and gcd(r, d) = 1.

The aim is to prove Proposition 1.5, to the effect that under these coprimality conditions, the morphism  $\tau_{\text{Hod}}(C; r, \bar{d}p)$  (10) is smooth. This smoothness is essential to the approach we take in this paper via vanishing/nearby cycle functors.

LEMMA 1.4 (Smoothness of  $M_{dR}$ ). The moduli space  $M_{dR}(C; r, \bar{d}p)$  of stable connections is nonempty, integral, quasi-projective, nonsingular, of the same dimension (11) of the corresponding moduli space  $M_{Dol}(C; r, \bar{d}p)$  of stable Higgs bundles of the same degree and rank. In particular, the fibers of the morphism  $\tau_{Hod}(C; r, \bar{d}p)$  (10) over the geometric points of  $\mathbb{A}^1_k$  are integral, nonsingular of the same dimension (11).

*Proof.* We drop some decorations. The fiber of  $\tau$  over the closed point 0 is  $M_{\text{Dol}}$ , and the fibers over the other closed points are isomorphic to  $M_{\text{dR}}$  in view of the trivialization (13). We are thus left with proving the assertions for the fiber  $M_{\text{dR}}$ .

Let  $C^{(1)}$  be the Frobenius twist of the curve C. Note that r and  $\overline{d} := d/p$  are also coprime. As recalled in §1.3, the moduli space  $M_{\text{Dol}}(C^{(1)}; r, \overline{d})$  is nonempty, integral, quasi-projective nonsingular of dimension (11). Since its dimension depends only on the genus  $g(C) = g(C^{(1)})$ of the curve C and on the rank r (cf. [Nit91, Proposition 7.4]), we have that  $M_{\text{Dol}}(C; r, \overline{d}p)$  and  $M_{\text{Dol}}(C^{(1)}; r, \overline{d})$  have the same dimension (11).

Let  $h_{\text{Dol}}(C^{(1)}, r, \overline{d}) \colon M_{\text{Dol}}(C^{(1)}, r, \overline{d}) \to A(C^{(1)}, \omega_{C^{(1)}}, r)$  be the Hitchin morphism for stable Higgs bundles for the canonical line bundle on  $C^{(1)}$ . Since stability and semistability coincide by coprimality, this Hitchin morphism is proper [Nit91, Theorem 6.1] and in fact projective since the domain is quasi-projective. Since the general fiber is connected, being the Jacobian of a nonsingular spectral curve [BNR89, Proposition 3.6], and the target is nonsingular, hence normal, this Hitchin morphism has connected fibers [Sta22, 03H0]. Being proper and dominant, it is also surjective.

Let  $h_{dR}(C; r, d): M_{dR}(C; r, d) \to A(C^{(1)}; r)$  be the de Rham-Hitchin morphism for stable connections on the curve C. This morphism is defined in [Gro16, Definition 3.16, p. 1007]. As seen in §1.3, it coincides with the specialization at t = 1 of the Hodge-Hitchin morphism  $h_{Hod}(C; r, d)$ .

By combining [Gro16, Theorem 1.1, Corollary 3.45 and Lemma 3.46], we see that the two morphisms  $h_{\text{Dol}}(C^{(1)}; r, \overline{d})$  and  $h_{\text{dR}}(C, r, \overline{d}p)$  are étale-locally equivalent over the base  $A(C^{(1)}; r)$ .

As noted in [Gro16, Corollary 3.47], this étale local equivalence implies that the de Rham– Hitchin morphism is proper and surjective. In fact, the de Rham–Hitchin morphism is projective in view of the quasi-projectivity of domain and target.

This étale local equivalence also implies that  $M_{dR}(C; r, \overline{d}p)$  is nonsingular of pure dimension dim  $M_{Dol}(C^{(1)}; r, \overline{d}) = \dim M_{Dol}(C; r, d)$  (11). By coupling the étale local equivalence with the connectedness of the fibers and with the integrality of  $M_{Dol}(C^{(1)}; r, \overline{d})$ , we deduce that  $M_{dR}(C, \overline{d}p)$  is integral as well.

PROPOSITION 1.5 (Smoothness of  $\tau_{\text{Hod}}: M_{\text{Hod}} \to \mathbb{A}^1_k$ ). The morphism  $\tau_{\text{Hod}}(C; r, \overline{d}p)$  (10) is a smooth fibration, that is, smooth, surjective, with connected fibers, onto the affine line  $\mathbb{A}^1_k$ . The Hodge moduli space  $M_{\text{Hod}}(C; r, \overline{d}p)$  of stable pairs is integral and nonsingular.

*Proof.* We drop some decorations. In particular, let us simply write  $\tau: M \to \mathbb{A}^1_k$ . Since the fibers of  $\tau$  are smooth (Lemma 1.4), in order to prove that  $\tau$  is smooth, it is enough to prove that  $\tau$  is flat. Once  $\tau$  is smooth, the smoothness and integrality of M follow from the flatness of  $\tau$  and the smoothness and integrality of the target and of the fibers of  $\tau$ .

We know that the fibers of  $\tau$  are nonsingular, integral and of dimension (11) (Lemma 1.4 and (16)). However, off the bat, we are unaware of an evident reason why M should be irreducible, or even reduced.

We know that  $\tau$  is flat over  $\mathbb{G}_{m,k} \subseteq \mathbb{A}_k^1$  by virtue of the trivialization (16). We need to verify that  $\tau$  is flat over the origin. This is a local question near the origin  $0 \in \mathbb{A}_k^1$ .

Let  $A := \operatorname{Spec}(k[x]_{(x)})$  (Hitchin bases, typically also denoted by A in this paper, do not appear in this proof) be the spectrum of the local ring of  $0 \in \mathbb{A}_k^1$ , and let  $\tau_A \colon M_A \to A$  be the base change of  $\tau$  via  $A \to \mathbb{A}_k^1$ . We need to show that  $M_A/A$  is flat.

The scheme  $M_A$  universally corepresents suitable equivalence classes of semistable *t*-connections on  $A \times C$ .

Note that  $\tau_A$  is surjective, hence dominant. Let 0 and  $\alpha$  be the closed and open points in A, respectively. Let  $(M_A)_0 = M_0$  and  $(M_A)_\alpha = M_\alpha$  be the corresponding fibers.

CLAIM 1.6. We have  $\overline{M_{\alpha}} \cap (M_A)_0 \neq \emptyset$ .

*Proof.* Let E be a rank r and degree  $\overline{dp}$  stable vector bundle on C (there are such bundles since their moduli space is an irreducible nonsingular variety of positive dimension one half of (11)). The stable bundle E is indecomposable [HL97, Corollary 1.2.8]. By [BS06, Proposition 3.1], the vector bundle E admits flat connections  $\nabla$ . Let  $(\mathcal{E}, x\nabla)$  be the *t*-connection on  $A \times C$  obtained by pulling back  $(E, \nabla)$  via the projection onto C and by twisting the connection by the function x on A. By [HL97, Proposition 1.3.7], we have that  $\mathcal{E}$ , being stable on the geometric fibers, is a stable bundle on  $A \times C$ , so that  $(\mathcal{E}, x\nabla)$  is a stable *t*-connection on  $A \times C$ . We thus have that  $(\mathcal{E}, x\nabla) \in M(A)$ . Then  $(E, (x = 0)\nabla) = (E, 0) \in M(k)$  is a specialization of the restriction of  $(\mathcal{E}, x\nabla)$  to the generic point of A.

CLAIM 1.7. We have  $(\overline{M_{\alpha}})_0 = (M_A)_0$ .

*Proof.* The closure  $\overline{M_{\alpha}}$  is integral, and it is a closed subscheme of  $M_A$ . It follows that the first fiber is a closed (and nonempty by Claim 1.6) subscheme of the integral nonsingular second fiber. By the upper-semicontinuity of the dimension of fibers at the source, the two fibers have the same dimension, hence they coincide by the integrality of the second fiber.  $\Box$ 

CLAIM 1.8. We have the equality of integral schemes  $\overline{M_{\alpha}} = M_{A, \text{red}}$ .

*Proof.* The first is a closed and dense (Claim 1.7 implies that they have the same geometric points) subscheme of the second, which is also integral.

Proof of Proposition 1.5, continued. By [Har77, §III.9.7], we have that  $M_{A,red} \to A$  and thus  $M_{red} \to A_k^1$  are flat.

It remains to show that  $M_A$  is indeed reduced: Let U be any nonempty affine open subset of  $M_A$ . Assume that  $f \in \Gamma(U, \mathcal{O}_U)$  is a nonzero nilpotent element, so that f maps to  $0 \in$  $\Gamma(U_{\text{red}}, \mathcal{O}_{U_{\text{red}}})$ . We have the factorization  $f = x^N g$ , where  $g \notin (x) \cdot \Gamma(U, \mathcal{O}_U)$ . By Claim 1.8, we have that  $U_{\text{red}}$  is integral. Therefore, either x or g is nilpotent in  $\Gamma(U, \mathcal{O}_U)$ . Since  $M_A \to A$ is dominant, we have that x is not nilpotent in  $\Gamma(U, \mathcal{O}_U)$ . Thus g is nilpotent. Since  $g \notin (x) \cdot$  $\Gamma(U, \mathcal{O}_U)$ , it maps to a nonzero nilpotent element in the special fiber of  $M_A$  over A, which contradicts the integrality of  $(M_A)_0$ .

# 1.5 Ad hoc proof of the properness of the Hodge–Hitchin morphism

The purpose of this section is to give an alternative proof of the properness of the Hodge–Hitchin morphism (Theorem 1.3) in the cases we need in this paper (Proposition 1.11). The proof is based on the application of the following rather general properness criterion and on the knowledge that the Hitchin and the de Rham–Hitchin morphisms are proper. In some sense, we collate these two properness statements. On the other hand, this collation does not seem to be immediate; see Remark 1.10. We are very grateful to Mircea Mustață for providing us with a proof of said criterion. We are also very grateful to Ravi Vakil for pointing out some counterexamples to some overly optimistic earlier versions of this criterion.

PROPOSITION 1.9 (An ad hoc properness criterion). Let  $m \circ f \colon X \to Y \to T$  be morphisms of schemes. We assume that

- (i) X is quasi-compact and quasi-separated, and Y is Noetherian;
- (ii) X and Y are integral, and Y is normal;
- (iii)  $f: X \to Y$  is separated, of finite type, surjective and with geometrically connected fibers;
- (iv) for every closed point  $t \in T$ , the morphism  $f_t \colon X_t \to Y_t$  obtained by base change is proper.

Then f is proper.

*Proof.* Let  $y \to Y$  be a closed point. The fiber  $f^{-1}(y) = X_y \to y$  is proper, as it is the fiber over y of the morphism  $f_t: X_t \to Y_t$ , with t := m(y). It follows that it is enough to prove the proposition when  $m: Y \to T$  is the identity morphism. We assume that we are in that case.

We have the commutative diagram



where (j, g) is a Nagata–Deligne completion [Con07] of the morphism f, that is, j is an open and dense immersion and g is proper; we can and do choose Z to be integral; (q, u) is the Stein factorization [Sta22, 03H0] of g, so that q has geometrically connected fibers and u is finite. Note that W is integral, that g and q are surjective and that u is finite and surjective.

By [dCHL18, Lemma 4.4.2] (this is stated for the case when W and Y are varieties over an algebraically closed field; however, the proof also works in our situation, where W is integral and Y is integral and Noetherian), there is a canonical factorization

$$u = s \circ i \colon W \xrightarrow{i} W' \xrightarrow{s} Y, \tag{18}$$

where i is finite radicial (hence a universal monomorphism) and surjective (hence a universal bijection), and s is finite, surjective, separable [dCHL18, Definition 4.4.1] and generically étale.

Our goal is to prove that u is bijective, that is, that s is bijective. If this were the case, then we would be done as follows. Since Y is quasi-compact and g is proper, by [Sta22, 04XU], we have that Z is quasi-compact. Therefore, the closed subspace  $Z \setminus X$  is also quasi-compact; thus, by [Sta22, 005E], if  $Z \setminus X$  is nonempty, then  $Z \setminus X$  has a closed point. Now let  $y \in Y$  be a closed point, and let  $w \in W$  be its unique pre-image via u. Then  $j(f^{-1}(y))$  is open in  $q^{-1}(w)$ , but it is also closed since  $X_y = f^{-1}(y)$  is proper over y by assumption. The connectedness of  $q^{-1}(w)$  implies that, set-theoretically,  $j(f^{-1}(y))$  equals  $q^{-1}(w)$ ; that is, j induces a bijection  $f^{-1}(y) \to g^{-1}(y)$ . Since this is true for every closed point  $y \in Y$  and since g is proper, we see that j induces a bijection between the closed points of Z and those of X. Therefore,  $Z \setminus X = \emptyset$ ; thus Z = X; that is, our contention that f is proper holds true.

We are left with proving that s is bijective. Note that the formation of the canonical factorization (18) is compatible with restrictions to open subsets in Y. Since W and Y are integral, Y is normal, s is finite, and a finite birational morphism from an integral scheme to an integral and normal scheme is an isomorphism [Sta22, 0AB1], it is enough to show that s is an isomorphism over a Zariski dense open subset U of Y. The remainder of the proof is dedicated to proving this assertion.

Note that h is dominant. Since the image Im(h) is constructible and dense, it contains a Zariski dense open subset  $V \subseteq W$ . Then  $u(W \setminus V)$  is a proper closed subset of Y with open and dense complement, which we denote by U. Then h is surjective over the open dense  $u^{-1}(U)$ . It follows that, in view of proving that s is an isomorphism, it is enough (as seen above) to prove it when h is surjective, which we assume hereafter.

For any closed point  $w \in W$ , by the connectedness of the fibers of f and the surjectivity of h, we have that, set-theoretically,  $h(f^{-1}(u(w)))$  is contained in the same connected component of  $u^{-1}(u(w))$  as w and also contains  $u^{-1}(u(w))$ . Therefore, as a scheme,  $u^{-1}(u(w))$  is connected, and it is finite over the residue field of u(w). By [Sta22, 00KJ], we have that  $u^{-1}(u(w))$ , as a set, is a singleton. We thus have that  $h^{-1}(w) = f^{-1}(u(w))$ . As seen above,  $j(h^{-1}(w))$  is then open and closed in the connected  $q^{-1}(w)$ .

As seen above, this implies that j is an isomorphism, and then f = g is proper with geometrically connected fibers. Since geometrically connected schemes are universally connected [Sta22,

054N], we have that s is separable and universally bijective. By [Sta22, 0154], we have that s is an isomorphism (recall that we shrunk Y to U). But then  $s: W' \to Y$  is an isomorphism over U, and this concludes our proof.

Remark 1.10. The case when f is the normalization of a nodal curve with a point removed from the domain and m is the identity shows that normality cannot be dropped from the list of assumptions in Proposition 1.9. The case when X the disjoint union of a line and a line without the origin, with f the natural morphism to a line with m the identity, shows that the irreducibility of X cannot be dropped. The case of f the square map  $\mathbb{G}_{m,k} \setminus \{-1\} \to \mathbb{G}_{m,k}$  (say char $(k) \neq 2$ ) and m the identity shows that the connectedness of the fibers cannot be dropped.

PROPOSITION 1.11 (Hodge–Hitchin is proper). Let C/k be a smooth curve over an algebraically closed field k of characteristic p > 0. Let  $\overline{d} \in \mathbb{Z}$ , and assume gcd  $(r, \overline{d}p) = 1$ . The Hodge–Hitchin morphism  $h_{\text{Hod}}(C; r, \overline{d}p)$  (13) is projective.

*Proof.* We drop some decorations. Since the domain and target are quasi-projective, it is enough to prove the properness of  $h_{\text{Hod}}$ . Recall (15) that for  $t = 0 \in \mathbb{A}^1_k$ , the morphism  $h_{\text{Hod},0}$  is the Hitchin morphism composed with the relative Frobenius  $\text{Fr}_A$  (a universal homeomorphism) of the Hitchin base. In view of (17), for  $t \in \mathbb{G}_{m,k}(k)$ , the morphism  $h_{\text{Hod},t}$  is isomorphic to the morphism  $h_{\text{dR}}$ .

We wish to apply Proposition 1.9 with  $m \circ f \colon X \to Y \to T$  given by  $\tau = \operatorname{pr}_{\mathbb{A}^1_k} \circ h_{\operatorname{Hod}}$  (13). In order to do so, we need to verify that the hypotheses (i)–(iv) are met in our setup.

Hypothesis (i) is clear. As to hypothesis (ii), we argue as follows. By Proposition 1.5, we have that  $X := M_{\text{Hod}}$  is integral and  $Y := A(C^{(1)}) \times \mathbb{A}^1_k$  is integral and normal (in fact nonsingular).

As to hypothesis (iii), we need to establish the surjectivity of  $f = h_{\text{Hod}}$  and the geometric connectedness of its fibers. The morphism  $h_{\text{Hod}}$  is surjective; in fact, according to the proof of Lemma 1.4, over the origin  $0 \in \mathbb{A}_k^1$ , the Hitchin morphism is surjective (and proper), and so is  $\text{Fr}_A$ ; over  $\mathbb{G}_{m,k}$ , the surjectivity follows from the trivialization (17) and the surjectivity of (the proper)  $h_{\text{dR}}$ .

Let us argue that the morphism  $h_{\text{Hod}}$  has geometrically connected fibers. It is enough to prove that for every closed point  $t \in \mathbb{A}^1_k$ , the morphism  $h_{\text{Hod},t}$  has geometrically connected fibers. In view of the trivialization (17), we need to prove this only for t = 0, where we get the Hitchin morphism composed with  $\text{Fr}_A$ , and for t = 1, where we get the de Rham–Hitchin morphism. The fibers of the Hitchin morphism are geometrically connected by Zariski's main theorem (and therefore so are the fibers of its composition with  $\text{Fr}_A$ ): the domain and target are nonsingular integral, and the general fibers are connected (Jacobians of nonsingular connected projective spectral curves; cf. [BNR89, Proposition 3.6]). As seen in the proof of Lemma 1.4, the fibers of the de Rham–Hitchin morphism for C in degree  $\overline{d}p$  are isomorphic to the fibers of the Hitchin morphism for the Frobenius twist  $C^{(1)}$  in degree  $\overline{d}$  and are thus also geometrically connected. This concludes the verification that hypothesis (iii) holds.

The morphisms  $f_t = h_{\text{Hod},t}$  are as follows: for t = 0 (15), the Hitchin morphism composed with  $\text{Fr}_A$ ; for t = 1 (14), the de Rham-Hitchin morphism; for  $t \neq 0$ , isomorphic to the de Rham-Hitchin morphism in view of the trivialization (17). The Hitchin morphism is proper [Fal93, Nit91, Sim94b]. The relative Frobenius morphism  $\text{Fr}_A$  is finite, hence proper. The de Rham-Hitchin morphism is proper by [Gro16, Corollary 3.47]. It follows that hypothesis (iv) holds as well.

We are now in the position to apply Proposition 1.9 and conclude.

#### 2. Cohomological Simpson correspondence in positive characteristic

Assumptions in § 2. In this section, we place ourselves in the following special case of the setup in § 1.3: C = C/k is a smooth curve over an algebraically closed field k of positive characteristic p > 0, the degree  $d = \overline{d}p$  is an integer multiple of the characteristic and gcd(r, d) = 1. At times, we drop some decorations.

The three main results in this section. We prove three main results, Theorems 2.1, 2.4 and 2.5. Theorem 2.1 is a canonical cohomological version of the Simpson correspondence between the moduli spaces of Higgs bundles and of connections. The perhaps surprising Theorem 2.4 yields a canonical isomorphism between the cohomology rings of the moduli space of connections and the moduli space of connections with nilpotent *p*-curvature tensor. The perhaps even more surprising, especially when compared with the well-known and evident "additive periodicity" (31), "*p*-multiplicative periodicity" Theorem 2.5 involves the Frobenius twists of a curve.

The perverse Leray filtrations we use. The étale cohomology ring  $H^*(M_{dR}(C), \overline{\mathbb{Q}}_{\ell})$  is filtered by the perverse Leray filtration  $P^{h_{dR}(C)}$  (6) associated with the de Rham-Hitchin morphism  $h_{dR}(C)$  (14). Similarly, we have the perverse Leray filtration  $P^{h_{Dol}(C)}$  (6) on  $H^*(M_{Dol}(C), \overline{\mathbb{Q}}_{\ell})$ associated with the Hitchin morphism  $h_{Dol}(C)$  (15).

Since the relative Frobenius morphism  $Fr_A$  in (15) is finite, in view of (7), we have that

$$P^{h_{\text{Dol}}(C)} = P^{h_{\text{Hod},0}(C)} \quad \text{on } H^*(M_{\text{Dol}}(C), \overline{\mathbb{Q}}_{\ell}) \,. \tag{19}$$

# 2.1 A cohomological Simpson correspondence in positive characteristic

Recall that the moduli space  $M_{dR}$  on the right-hand side of the forthcoming (20) is empty in characteristic zero. The  $M_{Dol}$  on the left-hand side is nonempty and lifts to characteristic zero.

THEOREM 2.1 (Cohomological Simpson correspondence,  $\operatorname{char}(k) = p > 0$ , I). Let C/k and  $\operatorname{gcd}(r, d = \overline{d}p) = 1$  be as in the beginning of §2 above.

There is a natural filtered isomorphism of cohomology rings

$$\left(H^*\left(M_{\rm Dol}(C;r,\overline{d}p),\overline{\mathbb{Q}}_{\ell}\right),P^{h_{\rm Dol}}\right)\simeq\left(H^*\left(M_{\rm dR}(C;r,\overline{d}p),\overline{\mathbb{Q}}_{\ell}\right),P^{h_{\rm dR}}\right).$$
(20)

*Proof.* We drop some decorations. Recall that the Hodge–Hitchin morphism at  $t = 1 \in \mathbb{A}_k^1$  coincides with the de Rham–Hitchin morphism, that is,  $h_{\text{Hod},1} = h_{dR}(C)$  (14), and that the Hodge–Hitchin morphism at  $t = 0 \in \mathbb{A}_k^1$  coincides with the composition of  $\text{Fr}_A \circ h_{\text{Dol}}$  (15). We apply the formalism of vanishing and nearby cycles recalled in §1.2 to the two morphisms

$$\tau \colon M_{\mathrm{Hod}}(C) \xrightarrow{h_{\mathrm{Hod}}} A(C^{(1)}) \times \mathbb{A}_{k}^{1} \xrightarrow{\pi := \mathrm{pr}_{2}} \mathbb{A}_{k}^{1} ,$$
$$\sigma \colon M_{\mathrm{dR}}(C) \times \mathbb{A}_{k}^{1} \xrightarrow{h_{\mathrm{dR}} \times \mathrm{Id}_{\mathbb{A}_{k}^{1}}} A(C^{(1)}) \times \mathbb{A}_{k}^{1} \xrightarrow{\pi := \mathrm{pr}_{2}} \mathbb{A}_{k}^{1} .$$

Note that the morphism  $\tau$  and  $\sigma$  share the second link  $\pi$ .

We take S to be a strict Henselianization of the spectrum of the completion of the local ring of the point  $i: 0 \to \mathbb{A}_k^1$ . By Lemma 1.4 and Proposition 1.5, the morphisms  $\sigma$  and  $\tau$  are smooth. In particular,  $\phi_\tau(\overline{\mathbb{Q}}_\ell) = 0$  and  $\phi_\sigma(\overline{\mathbb{Q}}_\ell) = 0$  (cf. Fact 1.2), so that we have  $\psi_\tau(\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$ on  $M_{\text{Dol}}(C) = M_{\text{Hod},0}(C)$  and  $\psi_\sigma(\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$  on  $M_{\text{dR}}(C)$ .

By Proposition 1.3, the morphism  $h_{\text{Hod}}$  is proper. Since the de Rham–Hitchin morphism  $h_{\text{dR}}$  is proper, the morphism  $h_{\text{dR}} \times \text{Id}_{\mathbb{A}^1_t}$  is proper. In particular, we have natural isomorphisms

in  $D_c^b(A(C^{(1)}) \times 0, \overline{\mathbb{Q}}_\ell)$  stemming from the proper base change isomorphisms  $(i^*h_* = h_*i^*, \psi h_* = h_*\psi)$ 

$$i^{*}h_{\mathrm{Hod}*}\overline{\mathbb{Q}}_{\ell} \simeq h_{\mathrm{Hod},0*}\overline{\mathbb{Q}}_{\ell}, \qquad i^{*}\left(h_{\mathrm{dR}} \times \mathrm{Id}_{\mathbb{A}^{1}_{k}}\right)_{*}\overline{\mathbb{Q}}_{\ell} \simeq h_{\mathrm{dR}*}\overline{\mathbb{Q}}_{\ell}, h_{\mathrm{Hod},0*}\overline{\mathbb{Q}}_{\ell} \simeq \psi_{\pi}\left((h_{\mathrm{Hod}}|_{\mathbb{G}_{m,k}})_{*}\overline{\mathbb{Q}}_{\ell}\right), \qquad h_{\mathrm{dR}*}\overline{\mathbb{Q}}_{\ell} \simeq \psi_{\pi}\left(h_{\mathrm{dR}} \times \mathrm{Id}_{\mathbb{G}_{m,k}}\right)_{*}\overline{\mathbb{Q}}_{\ell}.$$

$$(21)$$

By the trivializing isomorphism (17), we have a natural isomorphism between the two terms of type  $\psi_{\pi}$  in (21). We thus have a natural isomorphism in  $D_c^b(A(C^{(1)}))$ 

$$h_{\mathrm{Hod},0_*}\overline{\mathbb{Q}}_\ell \simeq h_{\mathrm{dR}*}\overline{\mathbb{Q}}_\ell.$$
 (22)

Ignoring the ring structure, the statement in cohomology follows by taking cohomology in (22); the filtered refinement follows from (19).

As to the ring structure, we argue as follows.

Recall that to obtain the isomorphism (22), we need to pass through three types of morphisms: firstly, the morphisms induced by  $i^* \to \psi$ ; secondly, the morphisms induced by the base change morphism; and lastly, the morphism induced by the trivializing isomorphism (17). We need to show that these three types of morphisms all preserve cup products.

We now consider the first type. Note that the vanishing cycle functor preserves cup products (see, for example, [III94, §4.3]). Upon taking cohomology on  $\mathbb{A}^1$ , the morphism  $i^* \to \psi$  induces the specialization morphism on stalks as defined in [Sta22, 0GJ2]. By the description of the specialization morphism in terms of pulling back sections via  $\bar{j}^*$  (8) as in [Sta22, 0GJ3], we see that the morphism  $i^* \to \psi$  preserves cup products.

To show that the second type of morphisms preserve cup products, we are reduced to showing that a base change morphism of the form  $i^*h_* \to h_*i^*$  preserves cup products. We can write the base change morphism as the composition  $i^*h_* \to i^*h_*i_*i^* \xrightarrow{\sim} i^*i_*h_*i^* \to h_*i^*$ , where the first morphism is induced by the unit morphism  $id \to i_*i^*$  and the last by the counit  $i^*i_* \to id$ . It is easy to check that both preserve cup products.

Finally, the trivializing isomorphism (17) is induced by an actual isomorphism (16) of varieties, and it does preserve cup products.

Remark 2.2 (Weights). If the curve C/k is obtained by extensions of scalars from a curve over a finite field, then the isomorphism (20) is compatible with the Frobenius weights (see [Del80, Theorem 6.1.13]). The same is also true for the isomorphisms in the forthcoming Theorems 2.4, 2.5, 3.6, 3.9, 3.11, 3.12 and 3.13.

#### 2.2 Cohomology ring of the space of connections with nilpotent *p*-curvature

The following Theorem 2.4 is a somewhat unexpected and surprising consequence of Theorem 2.1. This is because its analog (25) for the Dolbeault moduli space is well known to experts and proved using the  $\mathbb{G}_m$ -equivariance and properness of the Hitchin morphism, whereas in the de Rham case, there is no natural nontrivial  $\mathbb{G}_m$ -action. In particular, even ignoring the filtrations and the ring structure, there seems to be no clear a priori reason why the isomorphism (24) should hold additively.

The fiber  $N_{dR}$ . Let C/k and gcd  $(r, d = \overline{d}p) = 1$  be as in § 2. Let  $N_{dR}(C; r, \overline{d}p)$  be the fiber over the origin  $i_{o(1)}: o(1) \to A(C^{(1)}; r)$  of the de Rham–Hitchin morphism  $h_{dR}(C; r, d)$  (14). This is the moduli space of those stable connections of rank r and degree d with nilpotent p-curvature Higgs field. Let us drop r and d from the notation. The filtration  $P_{N_{dR}}$  on  $H^*(N_{dR}, \overline{\mathbb{Q}}_{\ell})$ . The inclusion of this fiber induces the cohomology ring homomorphism  $i_{o(1)}^* \colon H^*(M_{dR}) \to H^*(N_{dR})$ . The perverse *t*-structure on  $A(C^{(1)})$  induces a filtration P on the cohomology of the fiber  $\widetilde{M}_{dR}$  of  $h_{dR}$  over the strict localization  $\widetilde{o(1)}$  of o(1). By proper base change, restriction induces a cohomology ring isomorphism  $H^*(\widetilde{M}_{dR}, \overline{\mathbb{Q}}_{\ell}) \simeq$  $H^*(N_{dR}, \overline{\mathbb{Q}}_{\ell})$ , and, by transport of structure, the latter cohomology group inherits the filtration, denoted by  $P_{N_{dR}}$ , from the former (not to be confused with the perverse Leray filtration induced by the morphism  $N_{dR} \to o(1)$ , which is trivial-shifted by the degree in each cohomological degree). We thus have that restriction induces a filtered morphisms of cohomology rings

$$i_{o(1)}^* \colon \left(H^*\left(M_{\mathrm{dR}}\left(C; r, \overline{d}p\right), \overline{\mathbb{Q}}_{\ell}\right), P^{h_{\mathrm{dR}}}\right) \longrightarrow \left(H^*\left(N_{\mathrm{dR}}\left(C; r, \overline{d}p\right), \overline{\mathbb{Q}}_{\ell}\right), P_{N_{\mathrm{dR}}}\right).$$
(23)

Remark 2.3. The decomposition theorem [BBD82, Theorem 6.2.5] (stated over  $\mathbb{C}$ , but valid over any algebraically closed ground field) and the construction of  $P_{N_{dR}}$  imply that one can split the perverse filtrations  $P^{h_{dR}}$  and  $P_{N_{dR}}$  compatibly with the restriction morphism  $i^*_{o(1)}$ ; that is, this latter is a direct sum morphism for the two filtrations split into direct sums. In particular, if  $i^*_{o(1)}$ is an isomorphism, then it is a filtered isomorphism. Recall that isomorphisms that are filtered morphisms may fail to be filtered isomorphisms. By replacing "dR" with "Dol," we see that the same holds for  $P^{h_{Dol}}$  and  $P_{N_{Dol}}$ , where  $N_{Dol}$  is the fiber over  $o \in A(C)$  of the Hitchin morphism  $h_{Dol}: M_{Dol}(C) \to A(C)$ .

Recall our assumptions from §2: C/k, char(k) = p > 0 and gcd  $(r, d = \overline{d}p) = 1$ .

THEOREM 2.4 (The cohomology ring of  $N_{dR}$ ). The morphism (23) is a filtered isomorphism of cohomology rings

$$i_{o(1)}^* \colon \left(H^*\left(M_{\mathrm{dR}}\left(C; r, \overline{d}p\right), \overline{\mathbb{Q}}_{\ell}\right), P^{h_{\mathrm{dR}}}\right) \xrightarrow{\simeq} \left(H^*\left(N_{\mathrm{dR}}\left(C; r, \overline{d}p\right), \overline{\mathbb{Q}}_{\ell}\right), P_{N_{\mathrm{dR}}}\right).$$
(24)

*Proof.* We drop many decorations. We start by proving the forthcoming and seemingly well-known (cf., for example, [Hei15, Theorem 1]) (25), the proof of which remains valid without restrictions on rank, degree or characteristic of the ground field.

Let  $N_{\text{Dol}}$  be the fiber of the Hitchin morphism  $h_{\text{Dol}}: M_{\text{Dol}} \to A(C)$  over the origin  $i_o: o \to A(C)$ . The complex  $h_{\text{Dol}*}\overline{\mathbb{Q}}_{\ell M_{\text{Dol}}}$  is  $\mathbb{G}_m$ -equivariant for the natural  $\mathbb{G}_m$ -action on A(C) (cf. the paragraph following (15)). Since  $h_{\text{Dol}}$  is proper, proper base change (pbc), coupled with [dCMM18, Lemma 4.2], implies that the adjunction morphism

$$h_{\mathrm{Dol}*}\overline{\mathbb{Q}}_{\ell M_{\mathrm{Dol}}} \longrightarrow i_{o*}i_{o}^{*}h_{\mathrm{Dol}*}\overline{\mathbb{Q}}_{\ell M_{\mathrm{Dol}}} \xrightarrow{\simeq} i_{o*}h_{\mathrm{Dol}*}\overline{\mathbb{Q}}_{\ell N_{\mathrm{Dol}}}$$

induces an isomorphism. By taking cohomology, this morphism induces the restriction morphism in cohomology, which is thus an isomorphism of cohomology rings

$$i_o^*$$
:  $\left(H^*\left(M_{\text{Dol}}, \overline{\mathbb{Q}}_\ell\right), P^{h_{\text{Dol}}}\right) \xrightarrow{\simeq} \left(H^*\left(N_{\text{Dol}}, \overline{\mathbb{Q}}_\ell\right), P_{N_{\text{Dol}}}\right).$  (25)

In view of Remark 2.3, this is also a filtered isomorphism.

Recall diagrams (14) and (15). Let  $i_{o(1)}: o(1) \to A(C^{(1)})$  be the origin, so that  $N_{dR}$  is the corresponding fiber of the de Rham–Hitchin morphism  $h_{dR}: M_{dR} \to A(C^{(1)})$ . Let  $\operatorname{Fr}_{A(C)}^{-1}(o(1))$  be the fiber of  $\operatorname{Fr}_{A(C)}$  over o(1); it is supported at the origin  $o \in A(C)$ . The fiber  $h_{\text{Dol}}^{-1}(o) = N_{\text{Dol}}$  is a closed subscheme of the fiber  $[N_{\text{Dol}}] := h_{\text{Dol}}^{-1}(\operatorname{Fr}_{A(C)}^{-1}(o(1))) = h_{\text{Hod},0}^{-1}(o(1))$ , and these two fibers have the same reduced structure, hence the same cohomology ring (more precisely, identified

by pull-back). In view of the isomorphism (25), we have isomorphisms of cohomology rings  $H^*(M_{\text{Dol}}, \overline{\mathbb{Q}}_{\ell}) \simeq H^*([N_{\text{Dol}}], \overline{\mathbb{Q}}_{\ell}) \simeq H^*(N_{\text{Dol}}, \overline{\mathbb{Q}}_{\ell}).$ 

By applying the adjunction morphism of functors  $\mathrm{Id} \to i_{o(1)*}i_{o(1)}^*$  to the isomorphism (22), which we recall induces an isomorphism of cohomology rings, we obtain the following commutative diagram of morphisms of cohomology rings, where the vertical arrows are the restriction morphisms of cohomology rings, and with the indicated three isomorphisms of cohomology rings:

It follows that the fourth unmarked vertical arrow on the right-hand side, which is the restriction morphism  $i_{o(1)}^*$  in (24), is an isomorphism of cohomology rings.

Finally, since we now know that  $i^*_{o(1)}$  is an isomorphism and a filtered morphism (23), Remark 2.3 implies that  $i^*_{o(1)}$  is a filtered isomorphism, as predicated in (24).

# 2.3 Cohomology ring of moduli spaces for a curve and its Frobenius twist

Note that in the construction of the Frobenius twist  $C^{(1)} := C \times_k k$  of a k-scheme, we can replace the field automorphism  $fr_k \colon k \xrightarrow{\sim} k, a \mapsto a^p$  with any of its integer powers and obtain, for every integer  $n \in \mathbb{Z}$ , the *n*th iterated Frobenius twist  $C^{(n)}$  of C. The curve C and all its Frobenius twists have the same genus.

The following "multiplicative periodicity" result, involving the characteristic p as a factor and the Frobenius twists of C, is a simple, yet remarkable, consequence of Theorems 2.1 and 2.4 and [Gro16, Corollary 3.28]. It allows one to prove the forthcoming "multiplicative periodicity" result Theorem 3.11, involving only the curve C and not its Frobenius twists.

Recall our assumptions from §2: C/k, char(k) = p > 0 and gcd  $(r, d = \overline{d}p) = 1$ .

THEOREM 2.5 (*p*-Multiplicative periodicity with Frobenius twists). Let  $d = \tilde{d}p^m$ , with  $m \ge 0$  maximal. We have canonical isomorphisms of cohomology rings

$$H^*(M_{\text{Dol}}(C; r, \widetilde{d}p^m), \overline{\mathbb{Q}}_{\ell}) \cong H^*(M_{\text{Dol}}(C^{(m)}; r, \widetilde{d}), \overline{\mathbb{Q}}_{\ell}),$$
  
$$H^*(M_{\text{Dol}}(C^{(-m)}; r, \widetilde{d}p^m), \overline{\mathbb{Q}}_{\ell}) \cong H^*(M_{\text{Dol}}(C; r, \widetilde{d}), \overline{\mathbb{Q}}_{\ell});$$
(26)

we have the analogous isomorphisms when we replace  $\tilde{d}$  with  $\bar{d}$ .

These isomorphisms are filtered isomorphisms for the respective perverse Leray filtrations.

*Proof.* We prove the statements for d. The same line of argument applies to  $\overline{d}$ .

Since C can be any projective nonsingular curve of a fixed genus, by using Frobenius twists, we see that the two assertions are equivalent to each other. It is enough to prove the one in the top row. The case m = 0 is trivial. A simple induction on m shows that it is enough to prove the top row when m = 1.

We use the notation in the proof of Theorem 2.4. We recall that the two morphisms

$$h_{\text{Dol}}: M_{\text{Dol}}(C^{(1)}; r, \overline{d}) \to A(C^{(1)}; r) \text{ and } h_{\text{dR}}: M_{\text{dR}}(C; r, \overline{d}p) \to A(C^{(1)}; r)$$

are étale-locally equivalent over their common target  $A(C^{(1)}; r)$ ; see [Gro16, Corollary 3.28, Lemma 3.46]. This immediately implies that the two fibers over the origin  $N_{\text{Dol}}(C^{(1)}; r, \overline{d})$  and

 $N_{\mathrm{dR}}(C, \overline{d}p)$  are isomorphic as k-varieties. As in the proof of [Gro16, Corollary 3.45], we choose a distinguished isomorphism between  $h_{\mathrm{Dol}}$  and  $h_{\mathrm{dR}}$  over an étale neighborhood U over the origin of  $A(C^{(1)}, r)$ . By taking the fiber of this isomorphism over the origin of  $A(C^{(1)}, r)$ , we obtain a cohomology ring isomorphism  $\nu: H^*(N_{\mathrm{Dol}}) \xrightarrow{\sim} H^*(N_{\mathrm{dR}})$ . By the very construction of the filtrations  $P_{N_{\mathrm{Dol}}}$  and  $P_{N_{\mathrm{dR}}}$  in §2.2, the isomorphism  $\nu$  is filtered for  $P_{N_{\mathrm{Dol}}}$  and  $P_{N_{\mathrm{dR}}}$ .

By invoking the appropriate results in parentheses, we have the following chain of canonical ring filtered isomorphisms (filtrations are omitted for typographical reasons):

$$H^{*}(M_{\mathrm{Dol}}(C^{(1)},\widetilde{d}),\overline{\mathbb{Q}}_{\ell}) \stackrel{(25)}{\cong} H^{*}(N_{\mathrm{Dol}}(C^{(1)},\widetilde{d}),\overline{\mathbb{Q}}_{\ell})$$

$$\stackrel{[\mathrm{Grol6}, \S\S 3.28 \text{ and } 3.46]}{\cong} H^{*}(N_{\mathrm{dR}}(C,\widetilde{d}p),\overline{\mathbb{Q}}_{\ell})$$

$$\stackrel{(24)}{\cong} H^{*}(M_{\mathrm{dR}}(C,\widetilde{d}p),\overline{\mathbb{Q}}_{\ell})$$

$$\stackrel{(20)}{\cong} H^{*}(M_{\mathrm{Dol}}(C,\widetilde{d}p),\overline{\mathbb{Q}}_{\ell}).$$

This proves the top row in (26).

# 3. Cohomological equivalence of Hodge moduli spaces of curves

In §2, we worked with a fixed curve C/k over an algebraically closed field k of characteristic p > 0, and, under certain conditions on r, d and p, we have used the family  $\tau \colon M_{\text{Hod}}(C) \to \mathbb{A}_k^1$  to relate (the cohomology of)  $M_{\text{Dol}}$  and  $M_{\text{dR}}$  in the same degree (Theorem 2.1). We have also been able to relate  $M_{\text{Dol}}(C)$  and  $M_{\text{Dol}}(C^{(-n)})$  when the degrees differ by a factor  $p^n$  (*p*-multiplicative periodicity with Frobenius twists, Theorem 2.5).

In this section, we build on these results and, under certain conditions on r, d and p, we relate (the cohomology of)  $M_{\text{Dol}}$  with fixed degree for different curves of the same genus (Theorem 3.9), and with different degrees (Theorem 3.11) differing by a factor power of p for the same curve (hence for different curves of the same genus).

This latter result is then lifted to characteristic zero, where, coupled with the Dirichlet prime number theorem, it relates (the cohomology of)  $M_{\text{Dol}}$  in different degrees (Theorem 3.12) for a curve (hence for different curves). The existence of such an isomorphism in cohomology is known, but the compatibility of the perverse filtrations is new.

This result in characteristic zero is then specialized back to characteristic p > r (Theorem 3.13), where it is new.

The main technical tool employed in this section, which has not been used in proving the results in §2, is part of the compactification/specialization package developed in [dCa22] and generalized in part in [dC222]. We summarize what we need in Proposition 3.4. In order to have access to this package, we need to establish the smoothness (Proposition 3.1) and the properness (Proposition 3.3) of the morphisms we employ.

## 3.1 Relative moduli spaces: Smoothness and properness

In this subsection, we prove Proposition 3.1, namely, the smoothness of the Hodge moduli space  $M_{\text{Hod}}(C/B)$  for a projective smooth family C/B of curves over a nonsingular base curve B. We also prove Proposition 3.3, namely, the properness of the Hodge–Hitchin morphism for said family. These two results are the relative version over a base curve of Theorems 1.5 and 1.11. They are used in the proof of Theorem 3.9. In fact, we only need the specialization of these two

results to the case of the Dolbeault moduli space, where the properness of the Hitchin morphism is well known, while the smoothness assertion seems new, at least in positive characteristic.

PROPOSITION 3.1 (Smoothness of moduli over a base). Let C/B be a smooth curve over a reduced base B.

The following morphisms are smooth surjective and quasi-projective:

(i) 
$$\alpha_B \colon M_{\text{Hod}}(C/B, r, dp) \to B$$
, where  $\text{gcd}(r, d) = 1$ 

- (ii)  $\beta_B: M_{\text{Dol}}(C/B, r, d) \to 0_B \cong B$ , where gcd(r, d) = 1
- (iii)  $\tau_B \colon M_{\text{Hod}}(C/B, r, d) \to \mathbb{A}^1_B$ , where J is an algebraically closed field of characteristic p > 0and gcd  $(r, \overline{d}p) = 1$
- (iv)  $\gamma_B: M_{dR}(C/B, r, d) \to 1_B \cong B$ , where J is an algebraically closed field of characteristic p > 0 and gcd  $(r, \overline{d}p) = 1$ .

Moreover, if B is integral, then the domains of these morphisms are integral; if B is nonsingular, then the domains are nonsingular.

Proof. The surjectivity can be checked after base change via geometric points  $b \to B$ , in which case it follows from Proposition 1.5. The quasi-projectivity follows from the fact that the moduli spaces are quasi-projective over B. Note that parts (iii) and (iv) fail if we do not assume that dis a multiple of p, for then  $M_{dR}$  is empty. Part (i) implies parts (ii) and (iv) via the base changes  $0_B, 1_B \to \mathbb{A}^1_B$ . Part (i) coupled with the flatness of the morphisms  $\tau_b$  at the geometric points of B(Proposition 1.5) implies part (iii) in view of [GD66, 11.3.11], which states that a B-morphism  $f: X \to Y$  is flat if X is flat over B and the base change of f to each point  $b \in B$  is flat.

It follows that we only need to prove part (i). The proof of part (i) follows the same thread as the proof of the smoothness in Theorem 1.5. As the proof we are about to give shows, we are really implicitly proving part (ii) as we explicitly prove part (i).

Proof of part (i). Since the fibers of  $\alpha_B$  are smooth (Proposition 1.5), it is enough to prove the flatness of the locally finitely presented morphism  $\alpha_B$ . By the valuative criterion of flatness [GD66, 11.8.1], we can replace our *B* with the spectrum *A* of a DVR mapping to *B*. The proof that  $\alpha_A$  is flat is very similar to the proof of Proposition 1.5. Note that in order to use the valuative criterion of flatness, we need the assumption that *B* is reduced.

In the present context, the only point that requires a different proof is the analog of Claim 1.6 in the proof of Proposition 1.5. It is enough to exhibit a Higgs bundle on the curve  $X_A/A$  over the DVR A. In order to conclude the proof of part (i), it is thus sufficient to prove the forthcoming Claim 3.2. Let a and  $\alpha$  be the closed and open points of A.

CLAIM 3.2. We have  $\overline{M_{\alpha}} \cap M_a \neq \emptyset$ .

We have the Beauville–Narasimhan–Ramanan (BNR) correspondence [BNR89, Proposition 3.6] for smooth spectral curves: (a line bundle of the appropriate degree on a smooth degree r spectral curve S/A)  $\mapsto$  (a stable Higgs bundle of the appropriate degree on the curve  $X_A/A$ ).

If  $\overline{M_{\alpha}}$  and  $M_a$  were disjoint, then they would stay disjoint after any base change  $Z \to A$  covering a. It is thus enough to show that we can extend any line bundle on any smooth spectral curve  $S_a$  over  $C_a$  to a line bundle on a smooth spectral curve  $S_A$  over  $C_A$ , possibly after an étale base change  $Z \to A$  covering a.

Let  $u: S \to A(C_A/A, \omega_{C_A/A})$  be the universal spectral curve of degree r for the family  $C_A/A$ . Since the universal spectral curve is flat over the Hitchin base and the Hitchin base is flat over A, the universal curve is flat over A. By using the Jacobian criterion in connection with the polynomial expression for the equations of spectral curves, we see that S/A, being flat, is smooth. Then, since for every geometric point a on A, the fiber  $S_a$  is nonsingular integral, we see that S is integral. The morphism u is not smooth, but since general spectral curves are nonsingular—this is true over both points  $a, \alpha \in A$ —there is an open and dense subset  $U \subset A(C_A/A, \omega_{C_A/A})$ over which u is smooth and such that the resulting morphism  $U \to A$  is smooth and surjective. Moreover, the geometric fibers of S over U are nonsingular integral. By [Kle05, Theorem 9.4.8, Proposition 9.5.19], the Picard scheme  $\operatorname{Pic}_{S_U/U}$  exists as a smooth group scheme over U that is separated and locally of finite type over U. Note that  $\operatorname{Pic}_{S_U}/U$  is smooth and surjective. In particular,  $\operatorname{Pic}_{S_U}/A$  is smooth and surjective. By [Sta22, 054L], étale-locally over  $a \in A$ , the morphism  $\operatorname{Pic}_{S_U/U} \to A$  admits a section. Claim 3.2 is proved; part (i) and thus parts (ii), (iii) and (iv) follow.

Finally, since  $\alpha_B$ ,  $\beta_B$  and  $\gamma_B$  are smooth, their domains are nonsingular. By Lemmata 1.4 and 1.5, the fibers of  $\alpha_B$ ,  $\beta_B$  and  $\gamma_B$  are integral, in particular connected. Since moreover their images are connected, their domains must also be connected, thus integral.

PROPOSITION 3.3 (Properness of Hodge–Hitchin over a base). Let C/B be a smooth curve over a Noetherian integral and normal base B that is of finite type over an algebraically closed field of characteristic p > 0. Assume that  $d = \overline{d}p$  is a multiple of p and  $gcd(r, \overline{d}) = 1$ . The Hodge–Hitchin morphism  $h_{\text{Hod}}$  (13) is proper, in fact projective.

*Proof.* Since the Hodge–Hitchin morphism is quasi-projective, it is enough to prove that it is proper. To this end, it is enough to verify the hypotheses (i)–(iv) in the properness criterion 1.9, as has been done in the proof of Proposition 1.11. The verification is completely analogous.  $\Box$ 

## 3.2 Compactifications, vanishing cycles and specialization

Recall that if a family is not proper over a Henselian DVR (or, more geometrically, over a smooth curve), then the specialization morphism (9) is not necessarily defined, and, moreover, smoothness of the family alone is not sufficient in general to infer the vanishing we prove next. Such issues have been tackled over the complex numbers in [dCa22]. The discussion [dCZ22,  $\S$  5.1] shows that under favorable circumstances, we can apply the results in [dCa22], originally proved over the complex numbers, to a situation over an algebraically closed field, and over a DVR. Based on this, we state and prove the following.

PROPOSITION 3.4. (i) Let things be as in §2: C/k is a smooth curve over an algebraically closed field k, char(k) = p > 0 and gcd( $r, d = \overline{d}p$ ) = 1. Let  $\phi_{\tau}$  be the vanishing cycle functor associated with the morphism  $\tau_{\text{Hod}} \colon M_{\text{Hod}} \to \mathbb{A}_k^1$  (13) after base change  $S \to \mathbb{A}_k^1$  from a strict Henselianization of  $\mathbb{A}_k^1$  at the origin. We have the identity  $\phi_{\tau}(\tau_* \overline{\mathbb{Q}}_{\ell}) = 0$  for the vanishing cycles.

(ii) Let C/B be a smooth curve, where B is (the spectrum of) a strictly Henselian DVR. Assume gcd(r,d) = 1, and when the DVR is of mixed characteristic (0, p > 0), also assume p > r. The specialization morphism

$$H^*(M_{\text{Dol}}(C_s; r, d), \overline{\mathbb{Q}}_{\ell}) \xrightarrow{\text{sp}} H^*(M_{\text{Dol}}(C_{\overline{\eta}}; r, d), \overline{\mathbb{Q}}_{\ell})$$
(27)

is defined; it is a cohomology ring isomorphism and a filtered isomorphism for the perverse Leray filtrations induced by the respective Hitchin morphisms (12).

*Proof.* According to the discussion  $[dCZ22, \S5.1]$ , we can apply [dCa22, Lemma 4.3.3] (respectively, [dCa22, Theorem 4.4.2]) to the present situation (i) (respectively, (ii)), as long as the

morphism  $M_{\text{Hod}}(C/k) \to \mathbb{A}^1_k$  (respectively,  $M_{\text{Dol}}(C/B) \to B$ ) is smooth and the moduli space universally corepresents the appropriate functor. The smoothness has been proved in Proposition 3.1(iii) (respectively, Proposition 3.1(ii)), and, in view of the fact that stability equals semistability in the coprime case, the universal corepresentability in the coprime case is due to A. Langer [Lan14, Theorem 1.1]. This implies the desired conclusion (i) (respectively, (ii)).

Remark 3.5. If we replace the Dolbeault moduli spaces in Theorems 3.9, 3.12 and 3.13 with the moduli space of stable *L*-twisted Higgs bundles of degree coprime to the rank, where *L* either is the canonical bundle or satisfies deg  $L > \deg \omega_C$ , then we still have the conclusion analogous to that in Proposition 3.4(ii). This is because the analog of Proposition 3.1(ii) holds by the coprimality condition, with virtually the same proof.

## 3.3 Second proof of Theorem 2.1

In this section, we use Proposition 3.4 to give a second and simpler proof of Theorem 2.1. In fact, this proof yields an even stronger statement. On the other hand, the proof of Theorem 2.1 is more self-contained and, importantly, brings to the front the isomorphism (22), which plays a key role in the proof of Theorem 2.4, which is key to proving the *p*-multiplicative periodicity with Frobenius twists Theorem 2.5, which in turn plays a repeated role henceforth.

Recall our assumptions from §2: C/k, char(k) = p > 0 and gcd  $(r, d = \overline{d}p) = 1$ .

THEOREM 3.6 (Cohomological Simpson correspondence, char(k) = p > 0, II). The inclusions  $i_0: M_{\text{Dol}} \to M_{\text{Hod}}$  and  $i_1: M_{\text{dR}} \to M_{\text{Hod}}$  induce filtered isomorphisms of cohomology rings

$$H^*(M_{\text{Dol}}(C; r, \overline{d}p), \overline{\mathbb{Q}}_{\ell}) \xrightarrow{i_0^*} H^*(M_{\text{Hod}}(C; r, \overline{d}p), \overline{\mathbb{Q}}_{\ell}) \xrightarrow{i_1^*} H^*(M_{\text{dR}}(C; r, \overline{d}p), \overline{\mathbb{Q}}_{\ell})$$
(28)

for the perverse Leray filtrations associated with the Hitchin, the Hodge–Hitchin and the de Rham–Hitchin morphism, respectively.

*Proof.* By virtue of the smoothness of  $\tau_{\text{Hod}}$  (Theorem 1.5) and of the properness of the Hodge–Hitchin morphism (Theorem 1.11), we can apply Proposition 3.4, and we have  $\phi_{\tau}(\tau_*\overline{\mathbb{Q}}_{\ell}) = 0$ .

Since  $\phi_{\tau} := \phi_{\tau}[1]$  is t-exact for the perverse t-structure, we have the identity

$$\widetilde{\phi_{\tau}}(\,{}^{\mathfrak{p}}\mathcal{H}^{\bullet}(\tau_{*}\overline{\mathbb{Q}}_{\ell})) = \,{}^{\mathfrak{p}}\mathcal{H}^{\bullet}(\widetilde{\phi_{\tau}}(\tau_{*}\overline{\mathbb{Q}}_{\ell})) = 0$$

relating perverse cohomology sheaves. The local trivialization (16) implies that the restriction  ${}^{\mathfrak{P}}\mathcal{H}^{\bullet}(\tau_*\overline{\mathbb{Q}}_{\ell})_{|\mathbb{G}_{m,k}}$  is isomorphic to  $\mathscr{L}^{\bullet}[1]$ , where  $\mathscr{L}^{\bullet}$  is a suitably constant sheaf on  $\mathbb{G}_{m,k}$ .

By combining the two assertions of the previous paragraph with A. Beilinson's description of perverse sheaves via the vanishing cycle functor (see [Bei87, Proposition 3.1] or [dCM09, Theorem 5.7.7], for example), we see that the perverse cohomology sheaves  ${}^{\mathfrak{P}}\mathcal{H}^{\bullet}(\tau_*\overline{\mathbb{Q}}_{\ell})$  are constant sheaves shifted by [1].

A simple induction using the perverse truncation distinguished triangles, coupled with the fact that  $H^{\bullet \neq 0}(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell) = 0$ , shows that the complex  $\tau_* \overline{\mathbb{Q}}_\ell$  splits as the direct sum of its shifted perverse cohomology sheaves and thus, because they are shifts of constant sheaves, as the direct sum  $\bigoplus_{i \geq 0} R^i \tau_* \overline{\mathbb{Q}}_\ell[-i]$  of its shifted direct image sheaves, which, moreover, are constant sheaves of some rank.

The unfiltered assertion (28) follows. For the filtered version, we argue similarly, replacing  $\tau_* \overline{\mathbb{Q}}_{\ell}$  with the sequence of complexes  $\operatorname{pr}_{2*} {}^{\mathfrak{p}} \tau_{\leq \bullet} h_{\operatorname{Hod}*} \overline{\mathbb{Q}}_{\ell}$  (cf. (13)).

*Remark* 3.7. We can also prove Theorem 3.6, without using Beilinson's glueing of perverse sheaves, as follows.

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Since  $\phi_{\tau}(\tau_*\overline{\mathbb{Q}}_{\ell}) = 0$ , we have that  $R^i\tau_*\overline{\mathbb{Q}}_{\ell}$  is locally constant for each *i*. We also know that  $R^i\tau_*\overline{\mathbb{Q}}_{\ell}$  is constant over  $\mathbb{G}_m$ . Therefore, the local system  $R^i\tau_*\overline{\mathbb{Q}}_{\ell}$  is determined by a continuous representation  $\pi_1(\mathbb{A}^1_k, 1)$  of the étale fundamental group into  $\operatorname{GL}(H^i(M_{\mathrm{dR}}, \overline{\mathbb{Q}}_{\ell}))$  such that the composition with  $\pi_1(\mathbb{G}_m, 1) \to \pi_1(\mathbb{A}^1_k, 1)$  is trivial. Since the morphism  $\pi_1(\mathbb{G}_m, 1) \to \pi_1(\mathbb{A}^1_k, 1)$  is surjective [Sta22, 0BQI], the representation  $\pi_1(\mathbb{A}^1_k, 1) \to \operatorname{GL}(H^i(M_{\mathrm{dR}}, \overline{\mathbb{Q}}_{\ell}))$  is also trivial, so that  $R^i\tau_*\overline{\mathbb{Q}}_{\ell}$  is constant over  $\mathbb{A}^1_k$ .

*Remark* 3.8. If we disregard the filtrations, the ring isomorphisms (28) lift to Voevodsky motives: one combines the results [HP21, Theorem B1, Corollary B2 and the method of proof of Theorem 4.2] with the setup and smoothness results of this paper.

# 3.4 Cohomology ring of Dolbeault moduli spaces for two distinct curves

The goal of this subsection is to prove Theorem 3.9. Over the complex numbers, this theorem is an immediate consequence of the Simpson correspondence, for the two Dolbeault spaces have isomorphic Betti moduli spaces.

THEOREM 3.9 (Different curves, same degree). Let  $C_i/k$  be two smooth curves over an algebraically closed field. Assume that the rank and degree are coprime, gcd(r, d) = 1 (we do not assume that d is a multiple of p). There is a noncanonical isomorphism of cohomology rings that is a filtered isomorphism for the perverse Leray filtrations stemming from the respective Hitchin morphism

$$H^*\left(M_{\text{Dol}}(C_1; r, d), \overline{\mathbb{Q}}_\ell\right) \xrightarrow{\simeq} H^*\left(M_{\text{Dol}}(C_2; r, d), \overline{\mathbb{Q}}_\ell\right).$$
(29)

If, in addition, the ground field has characteristic p > 0 and  $d = \overline{d}p$  is an integer multiple of p, then we have a commutative diagram of isomorphisms of cohomology rings that are filtered isomorphisms for the respective perverse Leray filtrations:

*Proof.* The second statement (30) follows easily from the first one (29): we take the vertical isomorphisms in (29) to be the canonical ones of Theorem 2.1; we take (\*) to be that in (29); we close the diagram in the evident fashion.

We now construct the isomorphism (\*) in (29).

Let g be the genus of the curves  $C_1$ ,  $C_2$ . If g = 0, then the Dolbeault moduli spaces in question are a single point for r = 1 and empty for r > 1 [Nit91, § 7]; in either case, there is nothing left to prove. If g = 1, then we argue as in the forthcoming  $g \ge 2$  case, by using the irreducible moduli space of g = 1 curves with level structure [DR73, Corollary 5.6]. We may thus assume  $g \ge 2$ .

By the irreducibility assertion [DM69, §3] for the Hilbert scheme of tri-canonically embedded curves of genus  $g \ge 2$ , we can find a projective and smooth family C/B of genus g curves, with Ba nonsingular connected curve and with two closed fibers  $X_{b_i} \simeq C_i$ , where  $b_i \in B$  for i = 1, 2.

We conclude by taking (\*) to be (27) as in Proposition 3.4(ii) (triangulate  $b_1$  and  $b_2$  through a geometric generic point of B), which we can use in view of the smoothness assertion in Theorem 3.1(ii).

*Remark* 3.10. The conclusion (29) in Theorem 3.9 holds, with the same proof, in the setup of Remark 3.5. The key points are the properness of the Hitchin morphism in families [Fal93, Nit91, Sim94b] and the smoothness of the Dolbeault moduli space (the same proof as that of Proposition 3.1(ii) goes through).

## 3.5 *p*-Multiplicativity without Frobenius twist

The well-known additive periodicity of Dolbeault moduli spaces. Let C be a connected nonsingular projective curve over an algebraically closed field k. For arbitrary degree rank r and  $d \in \mathbb{Z}$ , there is a canonical isomorphism of cohomology rings for every  $n \in \mathbb{Z}$ 

$$H^*(M_{\text{Dol}}(C; r, d)) \simeq H^*(M_{\text{Dol}}(C; r, d + rn)).$$
 (31)

This follows from the fact that the choice of any degree n line bundle L on C induces, by the assignment  $(E, \phi) \mapsto (E \otimes L, 1_L \otimes \phi)$ , an isomorphism of Dolbeault moduli spaces that commutes with the Hitchin morphisms, hence induces a filtered isomorphism of cohomology rings as in (31). Since L can be made to vary in the connected  $\operatorname{Pic}^n(C)$ , the latter isomorphism is independent of the choice of  $L \in \operatorname{Pic}^n(C)$ .

We have the following consequence of Theorems 2.5 and 3.9 that came as a surprise to us. Note the very different nature of (32), that is, its expressing a periodicity under multiplication of the degree (coprime to the rank) by powers of p, when compared with (31), which expresses a periodicity when adding multiples of the rank to the degree.

The following result is concerned with the curve C only and should be compared with Theorem 2.5, which is concerned with a curve C and with its Frobenius twist  $C^{(1)}$ .

THEOREM 3.11 (*p*-Multiplicative periodicity without Frobenius twists). Let C/k be a smooth curve over an algebraically closed field k of characteristic p > 0. Assume gcd(r, d) = 1 (we do not assume that d is a multiple of p).

For every  $m \in \mathbb{Z}^{\geq 0}$ , there is a noncanonical isomorphism of cohomology rings

$$H^*(M_{\text{Dol}}(C;r,d)) \simeq H^*(M_{\text{Dol}}(C;r,dp^m))$$
(32)

that is a filtered isomorphism for the perverse Leray filtrations associated with the Hitchin morphism  $M_{\text{Dol}}(C) \to A(C)$ .

*Proof.* Combine Theorems 2.5 and 3.9, the latter with  $C_1 := C$  and  $C_2 := C^{(1)}$ .

#### 3.6 Cohomology ring of Dolbeault moduli spaces for two distinct degrees

In this section, we prove Theorems 3.12 and 3.13.

THEOREM 3.12 (Same curve, different degrees; char(k) = 0). Let C/k be a smooth curve over an algebraically closed field of characteristic zero. Fix the positive integer r (the rank). Let d, d'(the degrees) be any two integers coprime to r. There is a noncanonical ring isomorphism

$$H^*(M_{\text{Dol}}(C; r, d), \overline{\mathbb{Q}}_{\ell}) \cong H^*(M_{\text{Dol}}(C; r, d'), \overline{\mathbb{Q}}_{\ell})$$
(33)

that is a filtered isomorphism for the perverse Leray filtrations associated with the respective Hitchin morphisms.

*Proof.* Let  $a \in \mathbb{Z}$  be such that  $da \equiv d' \mod r$ . By the Dirichlet prime number theorem, there are infinitely many prime congruent to a modulo r. Choose any such prime p such that p > r and  $p \neq \ell$  ( $\ell$  as in  $\overline{\mathbb{Q}}_{\ell}$ ).

By the r-periodicity (31) and the p-multiplicativity (32), the statement of the theorem is true if we replace the characteristic zero algebraically closed ground field with any algebraically closed ground field of characteristic p.

By the Lefschetz principle, we can replace the given ground field with any algebraically closed field of characteristic zero, such as the forthcoming  $\kappa(\alpha)$ . In view of the isomorphisms (29), we can also replace the given curve C with any other curve of the same genus over  $\kappa(\alpha)$ , such as the forthcoming  $X_{\overline{\kappa(\alpha)}}$ .

Let A be the spectrum of a complete DVR of characteristic zero with algebraically closed residue field k of characteristic p. The content of this paragraph, namely that curves in positive characteristic can be lifted to characteristic zero, is standard and well known. For example, see [Obu12, Proposition 2.1]; see also [Mat11, Definition 4 and Theorem 5] and the remainder of that blog post. There is a smooth curve X/A, with closed special fiber  $X_a$  any prechosen integral nonsingular projective curve of genus g over  $\kappa(a)$ , and with generic geometric fiber  $X_{\overline{\kappa(\alpha)}}$  a curve of the same kind, but over the algebraically closed field  $\overline{\kappa(\alpha)}$  given by any chosen algebraic closure of the residue field  $\kappa(\alpha)$  of the generic point  $\alpha \in A$ .

By combining the characteristic p version of (33) with Proposition 3.4(ii), we get the following chain of cohomology ring isomorphisms, which are filtered isomorphisms for the respective perverse Leray filtrations (we drop the rank r):

$$H^*(M_{\text{Dol}}(X_{\overline{\kappa(\alpha)}};d)) \cong H^*(M_{\text{Dol}}(X_a;d)) \cong H^*(M_{\text{Dol}}(X_a;d')) \cong H^*(M_{\text{Dol}}(X_{\overline{\kappa(\alpha)}};d')).$$
  
The theorem is thus proved.

The theorem is thus proved.

Note that in the proof of Theorem 3.12 above, one can avoid using Proposition 3.4(ii) by spreading out C instead of lifting a chosen  $X_a$ . However, we use the lifting of  $X_a$  and Proposition 3.4(ii) in the proof of Theorem 3.13 below.

THEOREM 3.13 (Same curve, different degrees; char(k) = p > r). Let (r, d, d') be such that gcd(r,d) = gcd(r,d') = 1. Let C/k be a smooth curve over an algebraically closed field k of characteristic p > r. There is a noncanonical ring isomorphism

$$H^*(M_{\text{Dol}}(C, r, d), \overline{\mathbb{Q}}_{\ell}) \cong H^*(M_{\text{Dol}}(C, r, d'), \overline{\mathbb{Q}}_{\ell}),$$
(34)

which is a filtered isomorphism for the perverse Leray filtrations associated with the respective Hitchin morphisms.

*Proof.* Let X/A be a lift of C to characteristic zero as in the proof of Theorem 3.12. The desired conclusion in positive characteristic p follows by combining the analogous result (33) in characteristic zero with the specialization isomorphism (27). 

Note that Theorem 3.13 does not follow immediately by combining the *p*-multiplicativity (4) with the elementary periodicity (31) with respect to the rank. For example, take p = 3, r = 13, d' = 1, d = 15.

*Remark* 3.14. One can combine the results of Theorem 3.9 with those of Theorems 3.12 and 3.13 and obtain the evident "different curves, different degrees" version (omitted).

Remark 3.15 (Earlier results). (i) Point counts over finite fields, coupled with smoothness and purity arguments, give an equality of Betti numbers for the two sides of (33) and (34) over an algebraically closed ground field; see [GWZ20, Mel20, MS20, Sch16]. While such methods imply

the existence of an additive isomorphism preserving the perverse filtration, they do not seem to yield information on cup products.

(ii) Let the ground field be the complex numbers. If we replace  $M_{\text{Dol}}$  with the Betti moduli space  $M_B$ , then a well-known Galois-conjugation method yields a canonical isomorphism of cohomology rings analogous to (33). By nonabelian Hodge theory for gcd(r, d) = 1 over the complex numbers [HT04], we have cohomology ring isomorphisms  $H^*(M_B) \simeq H^*(M_{\text{Dol}})$ , so that we obtain a canonical cohomology ring isomorphism as in (33) but different from it. We are unaware of an evident reason why this canonical isomorphism should be compatible with the perverse filtration, the way (33) is. Added in revision: this issue is settled in the positive in [dCMSZ21].

(iii) Over a ground field of positive characteristic, given the lack of a Betti moduli space counterpart, the existence of a multiplicative (34) is new, and so is its compatibility with the perverse filtrations associated with the Hitchin morphisms.

#### Acknowledgements

We thank the referee for the excellent suggestions. We are very grateful to Mircea Mustață for providing us with a proof of the properness criterion afforded by Proposition 1.9. We are also very grateful to Michael Groechenig for many inspiring conversations on the subject. We thank Dan Abramovich, Barghav Bhatt, Hélène Esnault, Jochen Heinloth, Luc Illusie, Adrian Langer, Davesh Maulik, Junliang Shen, Ravi Vakil and Angelo Vistoli for very useful and pleasant email and Zoom exchanges. M. A. de Cataldo dedicates this paper to the memory of his parents, with love.

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